Modal Information Logic: decidability and completeness

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Extract of MSc thesis, supervised by Johan van Benthem and Nick Bezhanishvili

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Universiteit van Amsterdam

Plan for the talk

- Introducing the logics
- Stating the problems
- Outlining the strategy
- Solving the problems using the strategy

Definition (language and semantics)

The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \mathrm{sup} \rangle \varphi\psi,$$

and the semantics of '$\langle \mathrm{sup} \rangle$' is:

$$w \Vdash \langle \mathrm{sup} \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example

Definition (frames and logics)

Three classes of frames $(W, \leq)$, namely those where

- *(Pre)* $(W, \leq)$ is a preorder (refl., tr.);
- *(Pos)* $(W, \leq)$ is a poset (anti-sym. preorder); and
- *(Sem)* $(W, \leq)$ is a join-semilattice (poset w. all bin. joins)

Resulting in the logics MIL_{Pre}, MIL_{Pos}, MIL_{Sem}, respectively.

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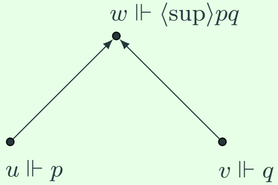

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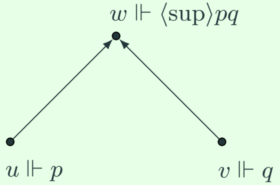

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Why MILs?

- Connect with other logics (e.g., truthmaker logics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}].

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

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Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

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Axiomatization (soundness and completeness)

MIL$_{Pre}$ is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

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Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

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Completeness of MIL_{Pre} (cont.)

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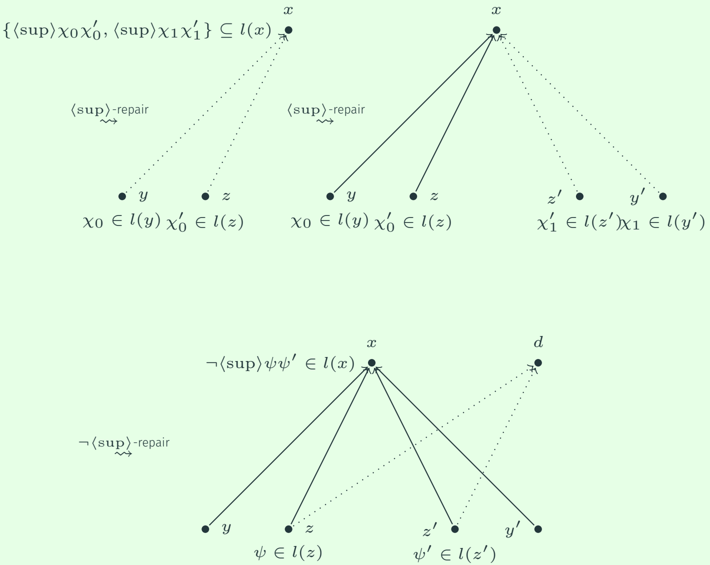

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(2) and (3): 'decidability via completeness'

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(3) Decidability through FMP on $\mathcal{C}$:

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  (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with 'semantically introduced' logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

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(3) **Decidability through FMP on $\mathcal{C}$:**

 (i) On $\mathcal{C}$, we get the FMP through filtration.

 (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

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  (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

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How about join-semilattices (i.e., MIL_{Sem})?

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Three ways to completeness (some intuitions for our proof):

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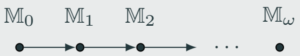

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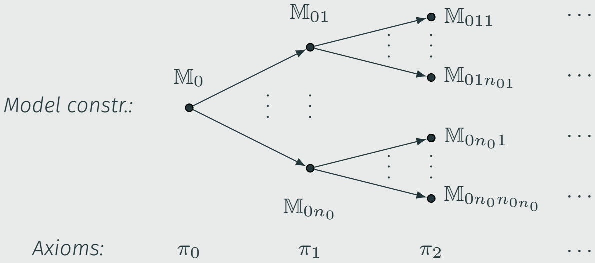

Thank you!

References I

Van Benthem, J. (1996). “Modal Logic as a Theory of Information”. In: *Logic and Reality. Essays on the Legacy of Arthur Prior.* Ed. by J. Copeland. Clarendon Press, Oxford, pp. 135–168 (cit. on pp. 6–11).

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Can we generalize these techniques?

MILs with informational implication '\'

(Natural) extensions of MIL_{Pre} and MIL_{Pos} [and $\mathbf{S4}$] are obtained by adding an informational implication '\'.

Definition

The language is given by adding '\' with semantics:

$$v \Vdash \varphi \backslash \psi \qquad \text{iff} \qquad \forall u, w([u \Vdash \varphi, w = \sup\{u, v\}] \Rightarrow w \Vdash \psi)$$

We denote the resulting logics as $MIL_{\backslash\text{-}Pre}, MIL_{\backslash\text{-}Pos}$, respectively.

The problems now become

(A\) axiomatizing $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$; and
(D\) proving (un)decidability.

The same (1)-(2)-(3) structure is used as before, but now we

(1') axiomatize the logic $\mathrm{Log}_{\backslash}(\mathcal{C})$;
(2') through representation show that $\mathrm{Log}_{\backslash}(\mathcal{C}) = MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$; and
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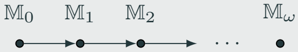

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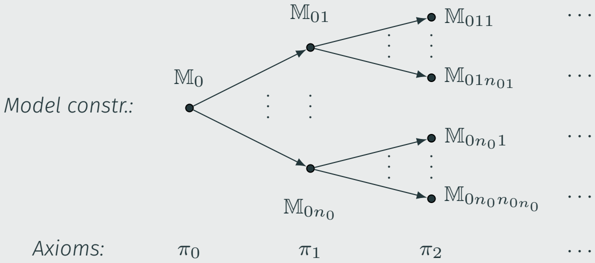

Conclusion and future work

What we have done:

- Thorougly surveyed the landscape of MILs on preorders and posets.
- Made crossings with the Lambek Calculus and truthmaker logics.[1]
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What comes next:

- Proving (un)decidability of MIL_{Sem} and solving the ancillary problems of fin. ax. and the FMP w.r.t. $\mathcal{C}_{Sem}$.
- Applying the techniques and heuristics of this thesis in other settings—not least those going into axiomatizing MIL_{Sem}.
- Further exploring how MILs connect to other logics.

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On ‘$\backslash$’ and ‘$\langle\mathrm{sup}\rangle$’

Example

Note how ‘$\langle\mathrm{sup}\rangle$’ and ‘$\backslash$’ are ‘inverses’:

$$\langle\mathrm{sup}\rangle p(p\backslash q) \rightarrow q$$

and

$$p \rightarrow q\backslash(\langle\mathrm{sup}\rangle pq)$$

are valid.